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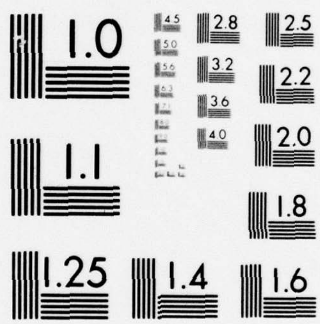

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BIFURCATION IN THE DUFFING EQUATION
WITH INDEPENDENT PARAMETERS, II

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BIFURCATION IN THE DUFFING EQUATION WITH INDEPENDENT PARAMETERS, II

Snopsis: In a previous paper, the authors gave a complete description of the number of even harmonic solutions of Duffing's equation without damping for the parameters varying in a full neighborhood of the origin in the parameter space. In this paper, the analysis is extended to the case of an independent small damping term. It is also shown that all solutions of the undamped equation are even functions of time.

1. Introduction.

Consider the Duffing equation with damping,

$$\frac{d^2u}{dt^2} + u = p_1 u + p_4 \frac{du}{dt} - p_2 u^3 + p_3 \cos t \tag{1.1}$$

where $p = (p_1, p_2, p_3, p_4)$ is a real four dimensional vector varying in a neighborhood U of the origin. Our objective is to discuss the 2π-periodic solutions of Equation (1.1) for each p in a sufficiently small neighborhood U. In a previous paper [2], the authors gave a complete description of the number of solutions for the undamped case, $p_4 = 0$, under the hypothesis that the solutions were continuous in p_3 and even in t. The hypothesis of evenness made it possible to reduce the discussion to a single bifurcation equation. When the damping term is present, such an hypothesis is meaningless and, therefore, two bifurcation equations must be considered. A complete analysis is possible after one has exploited the symmetry properties in Equation (1.1) to obtain detailed qualitative properties of the bifurcation equation. This latter information is also used to show that all 2π-periodic solutions of the undamped equation are even in t. The analysis in the present paper is in the same spirit as in [2] and uses ideas from [1], [3].

2. The Bifurcation Equations.

Consider the system:

$$\frac{d^2u}{dt^2} + u - p_1u + p_2u^3 = 0. \tag{2.1}$$

A 2π-periodic solution continuous in p_3, p_4 is a continuous function from a deleted neighborhood $V - \{(0,0)\}$ (depending on p_1, p_2) of $0 \in \mathscr{R}^2$ into the space of 2π-periodic functions which associates to each $(p_1, p_2) \in V - \{(0,0)\}$ a 2π-periodic solution $u(p_1,p_2,p_3,p_4)(t)$ of Equation (1.1). Furthermore, the set $\{u(p_1,p_2,p_3,p_4), (p_3,p_4) \in V - \{(0,0)\}\}$, with the uniform topology, is precompact and every limit point of this set as $(p_3,p_4) \to 0$ is a 2π-periodic solution of (2.1).

The idea for considering this particular type of continuous dependence on the parameters came for a paper of Hale and Taboas [4] in the consideration of 2π-periodic solutions of another type of equation. This definition is more general than the one considered by the authors in [2].

Since we are only interested in solutions of this form, it is necessary to discuss some detailed properties of the solutions of (2.1).

The ncessary information is contained in the following lemma, whose proof was given in [2].

Lemma 2.1. There is a constant $k > 0$ and a neighborhood U in $\mathscr{R}^2$ of $(p_1, p_2) = (0,0)$ such that a nonconstant 2π-periodic solution u of (2.1) exists for $(p_1, p_2) \varepsilon U$ if and only if $p_1 p_2 > 0$, this solution is unique except for a phase shift and satisfies

$$|u(t)| \leq k|p_1/p_2|^{1/2}.$$

If $(p_1, p_2) \varepsilon U$ and either $p_1 p_2 \leq 0$, $p_2 > 0$ or $p_1 \neq p_2 = 0$, the only 2π-periodic solution of (2.1) is $u = 0$. If $p_1 p_2 \neq 0$, $p_2 < 0$, there are the 2π-periodic solution $u = 0$, $u = \pm[(1-p_1)/|p_2|]^{1/2}$. If $p_1 = 0$, $p_2 = 0$, every solution of (2.1) is 2π-periodic.

For $p_2 < 0$, there are always two 2π-periodic solutions of Equation (1.1) which exist for $p = (p_1, p_2, p_3, p_4) \varepsilon \mathscr{R}^4$ in a neighborhood of zero and coincide with $\pm[(1-p_1/|p_2|]^{1/2}$ for $p_3 = p_4 = 0$. This is proved in the same manner as in [2] and is stated precisely in the following lemma.

Lemma 2.2. There is a neighborhood U in $\mathscr{R}^4$ of $p = 0$ and that for every $p \varepsilon U$, $p_2 < 0$, there exist two 2π-periodic solutions of Equation (1.1) which are continuous in (p_3, p_4) and coincide with $\pm[(1-p_1)/|p_2|]^{1/2}$ for $p_3 = p_4 = 0$. All other such 2π-periodic solutions of Equation (1.1) for $p_3 = p_4 = 0$ must coincide with the solution $u = 0$ of Equation (2.1) or a nonconstant periodic solution of Equation (2.1).

To obtain the other 2π-periodic solutions of Equation (1.1) continuous in (p_3,p_4), let $p_2 > 0$, $u = vp_2^{-1/2}$ in Equation (2.1) to obtain the equation

$$\frac{d^2v}{dt^2} + v = p_1 v + p_4 \frac{dv}{dt} - v^3 + \sigma \cos t \qquad (2.2)$$

where $\sigma = p_2^{1/2}p_3$. From Lemma 2.1, we know that $p_2^{1/2}u$ is bounded and, furthermore, that the only 2π-periodic solutions of Equation (2.2) that need to be considered are those for which v is small. If $p_2 < 0$, the same remark is true (of course, with $-v^3$ replaced by $+v^3$) provided the two solutions in Lemma 2.2 are excluded from the discussion.

We now discuss 2π-periodic solutions of Equation (2.2) for (v,p_1,σ,p_4) in a small neighborhood of the origin. The procedure will be the classical one used in [2] to obtain the bifurcation equations. Any 2π-periodic solution of Equation (2.2) for $p_1 = p_4 = \sigma = 0$ must be equal to $r\cos(t-\phi) + O(r^2)$ for some constants r,ϕ. Therefore, by letting $t \to t + \phi$, we will obtain a solution of our problem by considering the 2π-periodic solutions of the equation

$$\frac{d^2v}{dt^2} + v = p_1 v + p_4 \frac{dv}{dt} - v^3 + \sigma \cos t(t+\phi) \qquad (2.3)$$

which for $p_1 = p_4 = \sigma = 0$ are equal to $r \cos t + O(r^2)$.

Let $\mathscr{P} = \{h: \mathscr{R} \to \mathscr{R} : h$ is continuous $h(t+2\pi) = h(t)\}$ and for any $h \in \mathscr{P}$, let $|h| = \sup_t |h(t)|$. Let $P: \mathscr{P} \to \mathscr{P}$ be the projection defined by

$$(Ph)(t) = \frac{1}{\pi} \cos t \int_0^{2\pi} h(s) \cos s \, ds + \frac{1}{\pi} \sin t \int_0^{2\pi} h(s) \sin s \, ds. \quad (2.4)$$

For any $h \in \mathscr{P}$, the equation

$$\frac{d^2v}{dt^2} + v = h \quad (2.5)$$

has a solution in $\mathscr{P}$ if and only if $Ph = 0$. Furthermore, there is a continuous linear operator $K: (I-P)\mathscr{P} \to (I-P)\mathscr{P}$ such that $K(I-P)h$ is the unique solution of

$$\frac{d^2v}{dt^2} + v = (I-P)h \quad (2.6)$$

which satisfies $PK(I-P)h = 0$; that is, $K(I-P)h$ is simply the 2π-periodic solution of Equation (2.5) which does not contain cost, sint in its Fourier series. To this solution $K(I-P)h$, one can add an arbitrary linear combination of sint and cost to obtain the general solution of Equation (2.6). As remarked earlier, it is only necessary for us to add a term r cost.

If r is fixed and we define

$$P_r = \{h \in P: (Ph)(t) = r \cos t\},$$

$$f(v, p_1, p_4) = p_1 v + p_4 \frac{dv}{dt} - v^3$$

then v is a solution of Equation (2.3) in P_r if and only if

$$v = r\,\mathrm{cost} + w, \quad w \in (I-P)\mathscr{P} \tag{2.7a}$$

$$\frac{d^2w}{dt^2} + w = (I-P)f(r\cos(\cdot) + w, p_1, p_4) \tag{2.7b}$$

$$P[f(r\cos(\cdot) + w, p_1, p_4) + \sigma\cos(\cdot+\phi)] = 0 \tag{2.7c}$$

since $(I-P)\cos(\cdot+\phi) = 0$.

By an application of the Implicit Function Theorem, there are $\delta > 0$, $\varepsilon > 0$, such that, for $|r| < \delta$, $|p_1| + |p_4| < \varepsilon$, there is a unique solution $w^*(r,p_1,p_4)$ of Equation (2.7b) in $(I-P)\mathscr{P}$, the function $w^*(r,p_1,p_4)$ is analytic in r,p_1,p_4 and $w^*(0,p_1,p_4) = 0$. Furthermore, it is very easy to see that $w^*(r,p_1,0)$ is an even function of t since $v = r\,\mathrm{cost} + w$ and only powers of v occur on the right hand side of Equation (2.7b) for $p_4 = 0$.

Since $w^*(r,p_1,p_4)$ is uniquely determined it follows that there is a solution $v = r\,\mathrm{cost} + w \in P_r$ of Equation (2.3) which lies in a sufficiently small neighborhood of zero if and only if $v = r\,\mathrm{cost} + w^*(r,p_1,p_4)$ and (r,ϕ,p_1,σ,p_4) satisfy the bifurcation equations

$$P[f(r\cos(\cdot) + w^*(r,p_1,p_4),\ p_1,p_4) + \sigma\cos(\cdot + \phi)] = 0.$$

From the definition of P in Equation (2.4), these latter equations are equivalent to the system of equations

$$G_1(r,\phi,p_1,\sigma,p_4) \stackrel{\text{def}}{=} \sigma\cos\phi + \frac{1}{\pi}\int_0^{2\pi} f(r\cos t + w^*(r,p_1,p_4)(t),p_1,p_4)\cos t\,dt = 0$$

$$G_2(r,\phi,p_1,\sigma,p_4) \stackrel{\text{def}}{=} \sigma\sin\phi + \frac{1}{\pi}\int_0^{2\pi} f(r\cos t + w^*(r,p_1,p_4)(t),p_1,p_4)\sin t\,dt = 0.$$

Since $w^*(r,p_1,0)$ is an even function, it is clear that $G_2(r,\phi,p_1,\sigma,0) = \sigma\sin\phi$. For $w^*(r,p_1,p_4) = 0$, it is easy to evaluate the above integrals. If this computation is made and one uses the Taylor series to obtain the order estimate $0(|p_1r| + |p_4r| + |r|^3)$ on $w^*(r,p_1,p_4)$, the bifurcation equations become

$$G_1(r,\phi,p_1,\sigma,p_4) = \sigma\cos\phi + p_1r - \frac{3}{4}r^3 + rg_1(r,p_1,p_4) = 0 \tag{2.8a}$$

$$G_2(r,\phi,p_1,\sigma,p_4) = \sigma\sin\phi + p_4r + p_4rg_2(r,p_1,p_4) = 0 \tag{2.8b}$$

where

$$\begin{aligned} g_1(r,p_1,p_4) &= 0(|p_1|^2 + |p_4|^2 + |p_1p_4| + r^2|p_1| + r^2|p_4| + r^4) \\ g_2(r,p_1,p_4) &= 0(r^2 + |p_1| + |p_4|). \end{aligned} \tag{2.9}$$

These results are summarized in the following lemma.

Lemma 2.3. _There is a neighborhood_ $U \subset \mathscr{R}^4$ _of_ $(r,p_1,\sigma,p_4) = 0$ _and a neighborhood_ $V \subset \mathscr{P}$ _of_ $v = 0$ _such that Equation (2.3) has a solution_ $v \in P_r \cap V$ _for_ $(r,p_1,\sigma,p_4) \in U$ _and a given_ ϕ _if and only if_ (r,p_1,σ,p_4,ϕ) _satisfy the bifurcation Equations (2.8) where_ g_1,g_2 _satisfy_ (2.9).

An immediate corollary is the following result on the undamped Duffing equation.

Corollary 2.1. *There is a neighborhood* $U \subset \mathscr{R}^2$ *of* $(p_1, \sigma) = 0$ *and a neighborhood* $V \subset \mathscr{P}$ *of* $v = 0$ *such that the only* 2π*-periodic* $\mathscr{P}$ *solutions in* V *of the undamped Duffing equation*

$$\frac{d^2v}{dt^2} + v = p_1 v - v^3 + \sigma \cos t \tag{2.10}$$

are even functions of t *if* $\sigma \neq 0$.

Proof. For $p_4 = 0$, the second bifurcation Equation (2.8b) is $\sigma \sin \phi = 0$. Therefore, if $\sigma \neq 0$, we must have $\phi = 0$. If $\phi = 0$, then $v(t) = r \cos t + w^*(r, p_1, 0)(t)$ is an even solution of Equation (2.10). Since all solutions of Equation (2.10) can be obtained from Equation (2.3) and the Lemma 2.3, we have proved the desired result.

Remark 2.1. The conclusion of Corollary 2.1 is also valid for the equation

$$\frac{d^2v}{dt^2} + v = p_1 v - v^3 + \sigma f(t)$$

where $f(t)$ is even in t and $\int_0^{2\pi} f(t) \cos t \, dt = \pi$. To prove this, one considers the same equation with t replaced by $t + \phi$ and observes that the bifurcation equation obtained by projecting onto $\sin t$ has the form $\sigma(\sin \phi) h(r, \phi, p_1) = 0$ with $h(0, \phi, 0) = 1$.

One can now repeat the same argument as in the proof of Corollary 2.1 to complete the proof.

3. Analysis of the Bifurcation Equations.

In this section, we analyze the bifurcation Equations (2.8) for (r,ϕ) and determine the surfaces (bifurcation surfaces) in the parameter space across which the number of solutions (r,ϕ) changes. If the number of solutions is to change at some point in the parameter space, then there must be a multiple root (r,ϕ) of Equations (2.8) at these values of the parameters; that is, $\partial(G_1,G_2)/\partial(r,\phi) = 0$. Our objective, therefore is to obtain the values of the parameters at which multiple solutions of Equations (2.8) occur. Basic to the investigation is the following a priori estimate.

Lemma 3.1. *There is a neighborhood* $U \subset \mathscr{R}^4$ *of the point* $(r,p_1,\sigma,p_4) = (0,0,0,0)$ *and a constant* c *such that all solutions* (r,ϕ) *of Equations (2.8) with* $(r,p_1,\sigma,p_4) \in U$ *satisfy*

$$|r| \le c(|p_1|^{1/2} + |\sigma|^{1/3} + |p_4|^{1/2}).$$

Proof. If this is not the case, then there is a sequence $(\phi_n,r_n,p_{1n},\sigma_n,p_{4n})$ with $(r_n,p_{1n},\sigma_n,p_{4n}) \to (0,0,0,0)$ as $n \to \infty$ such that

$$\frac{|p_{1n}|^{1/2}}{r_n} \to 0, \quad \frac{|\sigma_n|^{1/3}}{r_n} \to 0, \quad \frac{|p_{4n}|^{1/2}}{r_n} \to 0,$$

as $n \to \infty$. Dividing the Equation (2.8a) by r_n^3 and letting $n \to \infty$, this implies that $0 = -\frac{3}{4}$, which is a contradiction. This proves the lemma.

Lemma 3.1 justifies the scaling

$$r = \sigma^{1/3}\rho, \quad p_1 = p\sigma^{2/3}, \quad p_4 = \delta\sigma^{2/3}$$

in Equations (2.8). Making this change of variables and dividing the equation by σ, one obtains the equivalent equations for $\sigma \neq 0$.

$$\overline{G}_1 \overset{\text{def}}{=} p\rho - \frac{3}{4}\rho^3 + \cos\phi + \rho 0(\sigma^{2/3}) = 0$$

$$\overline{G}_2 \overset{\text{def}}{=} \delta\rho + \sin\phi + \delta\rho(\sigma^{2/3}) = 0.$$

For $(p,\sigma,\delta) = (0,0,0)$, these equations only have the solution $(\rho,\phi) = ((\frac{4}{3})^{1/3},0)$. Furthermore,

$$\frac{\partial(\overline{G}_1,\overline{G}_2)}{\partial(\rho,\phi)} = \begin{bmatrix} -\frac{9}{4}\left(\frac{4}{3}\right)^{1/3} & 0 \\ 0 & 1 \end{bmatrix}$$

at these points. The Implicit Function Theorem implies there are $\varepsilon_0 > 0$ and $\alpha \geq 1$ such that Equations (2.8) have unique solutions near the points (ρ,ϕ) above for $|\sigma| \leq \varepsilon_0$, $|p|^3 + |\delta|^3 \leq 1/\alpha$. Therefore, in the original variables (p_1,σ,p_4), no bifurcation occurs in the region

$$S_{\alpha,\varepsilon_0} = \{(p_1,\sigma,p_4): |\sigma| \leq \varepsilon_0, |p_1|^3 + |p_4|^3 \leq \sigma^2/\alpha\}$$

This implies that the bifurcation surfaces in an ε_0 neighborhood of $(p_1,\sigma,p_4) = 0$ must be in the region

$$R_{\alpha,\varepsilon_0} = \{(p_1,\sigma,p_4): p_1^2 + p_4^2 \le \varepsilon_0^2,\ \sigma^2 \le \alpha(|p_1|^3 + |p_4|^3)\}.$$

In the region

$$R_1 = R_{\alpha,\varepsilon_0} \cap \{(p_1,\sigma,p_4): p_4 > 0,\ |p_1| \le p_4\}$$

the scaling

$$r = \mu\rho,\ p_4 = \mu^2,\ p_1 = n\mu^2,\ \sigma = \nu\mu^3,\ n \in [-1,1],\ |\nu| \le \alpha(1 + |n|)$$

leads to the new bifurcation equations

$$\begin{aligned} \overline{G}_1 &= n\rho - \frac{3}{4}\rho^3 + \nu\cos\phi + \rho 0(\mu^2) = 0 \\ \overline{G}_2 &= \rho + \nu\sin\phi + \rho 0(\mu^2) = 0. \end{aligned} \tag{3.1}$$

For $\mu = 0$, one obtains directly from Equation (3.1) that

$$\Delta \overset{\text{def}}{=} \det \frac{\partial(\overline{G}_1,\overline{G}_2)}{\partial(\rho,\phi)} = -\rho\left[(n - \frac{9}{4}\rho^2) - 1\right]$$

for any solution (ρ,ϕ) of (3.1). Since $n^2 \le 1$, the only solution of $\Delta = 0$ is $\rho = 0$. But this implies $\nu = 0$ which implies the Equation (2.3) is autonomous. For this situation, we know Equation (2.3) has only the solution $v = 0$ for any $p_4 > 0$. Therefore, no bifurcation occurs in R_1.

One can make a similar analysis in the region $R_{\alpha,\varepsilon_0} \cap \{(p_1,\sigma,p_4): p_4 < 0, |p_1| \leq -p_4\}$ and arrive at the same conclusion.

Consider now the region

$$R_2 = R_{\alpha,\varepsilon_0} \cap (p_1,\sigma,p_4): p_1 < 0, |p_4| \leq -p_1\}.$$

If we use the scaling $r = \mu\rho$, $p_1 = -\mu^2$, $\sigma = \nu\mu^3$, $p^4 = m\mu^2$, $|m| \leq 1$, $|\nu| \leq \alpha(1 + |m|)$ the bifurcation equations become

$$\bar{G}_1 = -\rho - \frac{3}{4}\rho^3 + \nu \cos \phi + 0(\mu^2)$$

$$\bar{G}_2 = m\rho + \nu \sin \phi + 0(\mu^2)$$

and, at $\mu = 0$,

$$\det \frac{\partial(\bar{G}_1,\bar{G}_2)}{\partial(\phi,\rho)} = \rho[-(1 + \frac{9}{4}m^2)(1 + \frac{3}{4}\rho^2) - m^2].$$

This quantity is zero only if $\rho = 0$. With an argument similar to the one for R_1, one observes there is no bifurcation at this point.

Let us now analyze the region

$$R_3 = R_{\alpha,\varepsilon_0} \cap \{(p_1,\sigma,p_4): p_1 > 0, |p_4| \leq p_1\}.$$

For the scaling

$$r = \mu\rho,\ p_1 = \mu^2,\ \sigma = \nu\mu^3,\ p_4 = m\mu^2,\ |m| \le 1,\ |\nu| \le \alpha(1 + |m|)$$

the bifurcation equations become

$$\begin{aligned} \bar{G}_1 &= \rho - \frac{3}{4}\rho^3 + \nu \cos\phi + \rho O(\mu^2) = 0 \\ \bar{G}_2 &= \nu \sin\phi + m\rho + m\rho O(\mu^2) = 0. \end{aligned} \tag{3.2}$$

Recall that we need only consider these equations for $|m| \le 1$, $|\nu| \le \alpha(1 + |m|)$ and ρ small. Therefore, ρ will be in a bounded set, $|\rho| \le 2c(1+\alpha^{1/3})$. The corresponding determinant of the Jacobian matrix is given by

$$\bar{\Delta} = \det \frac{\partial(\bar{G}_1, \bar{G}_2)}{\partial(\rho,\phi)} = \det \begin{bmatrix} 1 - \frac{9}{4}\rho^2 + O(\mu^2) & -\nu \sin\phi \\ m + mO(\mu^2) & \nu\cos\phi \end{bmatrix}$$

$$= \nu\left[\left(1 - \frac{9}{4}\rho^2\right)\cos\phi + m\sin\phi + O(\mu^2)\right].$$

For $\nu \neq 0$ (i.e. $\sigma \neq 0$), the multiple solutions of our scaled bifurcation equations are the solutions of the three equations

$$\begin{aligned} h_1 &\overset{\text{def}}{=} \rho - \frac{3}{4}\rho^3 + \nu\cos\phi = \rho O(\mu^2) \\ h_2 &\overset{\text{def}}{=} m\rho + \nu\sin\phi = m\rho O(\xi^2) \\ h_3 &\overset{\text{def}}{=} (1 - \frac{9}{4}\rho^2)\cos\phi + m\sin\phi = O(\mu^2). \end{aligned} \tag{3.3}$$

The outline of the program is now as follows.

Treating ρ as a parameter, we determine ϕ,ν,m as functions of ρ such that Equations (3.3) for $\mu = 0$ are satisfied. If $\det \partial(h_1,h_2,h_3)/\partial(\phi,\nu,m) \neq 0$, then we can determine solutions of Equations (3.3) for μ sufficiently small. Elimination of the parameter ρ from the resulting solutions $\nu^*(\rho,\mu)$, $m^*(\rho,\mu)$ will give the possible bifurcation surfaces ν as a function of m, or perhaps m as a function of ν,μ. We then verify that the number of solutions (ρ,ϕ) of Equations (3.2) [or equivalently, the number of solutions (r,ϕ) of Equation (2.8)] changes by two as this surface is crossed. This proves that it is a bifurcation surface.

If $h_1 = h_2 = h_3 = 0$, $\mu = 0$ then

$$\cot \phi = \frac{1 - \frac{3}{4}\rho^2}{m} = \frac{-m}{1 - \frac{9}{4}\rho^2} \tag{3.4}$$

and so

$$m^2 = \left(\frac{9}{4}\rho^2 - 1\right)\left(1 - \frac{3}{4}\rho^2\right) \tag{3.5}$$

also, $h_1 = h_2 = 0$ further implies

$$\nu^2 = \rho\left(\frac{3}{4}\rho^2 - 1\right)^2 + m^2\rho^2 \tag{3.6}$$

Equations (3.4), (3.5), (3.6) are the parametric forms of the solutions ϕ,m,ν as functions of ρ if we substitute Equation (3.5) into Equations (3.4) and (3.6). Eliminating ρ from Equation (3.5), (3.6), we have

$$\nu^2 = \frac{8}{81}\,[1 + 9m^2 \pm (1-3m^2)^{3/2}]. \qquad (3.7)$$

The locus of the points in the (ν,m) plane described by Equation (3.7) are plotted in Figure 1.

If

$$\Delta_1 = \det \partial(h_1,h_2,h_3)/\partial(\phi,m,\nu),$$

then

$$\Delta_1 = \nu \sin\phi - \rho\left(\frac{9}{4}\rho^2 - 1\right)\cos\phi \sin\phi - m\rho\cos^2\phi.$$

As remarked earlier, if $\Delta_1 \neq 0$ at a solution $(\phi_0,m_0,\nu_0,\rho_0)$ of $h_1 = h_2 = h_3 = 0$, then we can obtain a solution (ϕ,m,ν) of Equations (3.3) for $|\rho-\rho_0|$ and $|\mu|$ sufficiently small by employing the Implicit Function Theorem.

If $h_1 = h_2 = h_3 = 0$, then

$$\Delta_1 = -2m\sqrt{\frac{9}{4}\,(2\pm\sqrt{1-3m^2})}$$

and $\Delta_1 = 0$ implies $m = 0$ since $m^2 \leq 1$. If $m = 0$, then $(\nu^2,\rho^2,\phi) = (16/81,\ 4/9,\ 0)$ or $(0,\ 4/3,\ \pi/2)$.

Therefore, at each solution $(\phi_0,m_0,\nu_0,\rho_0)$ of $h_1 = h_2 = h_3 = 0$ for which $(\nu_0,m_0) \neq (\pm 4/9,0)$ or $(0,0)$, there is an $\varepsilon > 0$ and unique solutions $\phi(\rho,\mu),m(\rho,\mu)\nu(\rho,\mu)$ of Equations (3.3) for

$|\rho-\rho_0|$, $|\mu| < \varepsilon$, $\phi(\rho_0,0) = \rho_0$, $m(\rho_0,0) = m_0$, $\nu(\rho_0,0) = \nu_0$. Thus, if we exclude a neighborhood V of the points $(\pm 4/9,0)$ and $(0,0)$ in the (ν,m) plane, then we can find a $\mu_0 > 0$ such that the Equations (3.3) can be uniquely solved for (ν,m,ν) as functions of (ρ,μ) for $|\mu| \le \mu_0$ and $|\rho| \le 2c(1+\alpha^{1/3})$, the a priori bound on ρ. Elimination of ρ from the corresponding functions m,ν gives part of the surface indicated in Figure 2 in terms of the unscaled variables (p_1,σ,p_4). The first approximation to the explicit formula for this surface is obtained from the scaling and Equation (3.7) and is given by

$$\sigma^2 = \frac{8}{81}\left[p_1^3 + 9p_1p_4^2 \pm (p_1^2 - 3p_4^2)^{3/2}\right] \tag{3.8}$$

At the points $(\nu,m) = (\pm 4/9,0)$ in the (ν,m) plane where $\Delta_1 = 0$, we compute another Jacobian and apply the implicit function theorem. At these points and for $(\rho,\phi) = (\pm 2/3,0)$, we see that

$$\Delta_2 = \det \frac{\partial(h_1,h_2,h_3)}{\partial(\rho,m,\nu)} = -2 \neq 0.$$

Therefore, we can solve the Equations (3.3) for ρ,m,ν as functions of ϕ,μ for ϕ,μ sufficiently small. Eliminating the parameter ϕ gives the bifurcation surface near the points $(\pm 4/9,0)$.

At the point $(\nu,m) = (0,0)$, the corresponding solutions (ρ,ϕ) of $h_1 = h_2 = h_3 = 0$ is $(\pm 2/\sqrt{3},\pi/2)$. If we let $\nu = \alpha m$, then the bifurcation equations are

$$h_1 = \rho - \frac{3}{4}\rho^3 + \alpha m \cos\phi = 0(\mu^2)$$
$$\frac{h_2}{m} = \rho + \alpha \sin\phi = 0(\mu^2). \tag{3.9}$$

For (ρ,ϕ) either of the above points, we have $h_2/m = 0$ implies $\alpha = \pm 2/\sqrt{3}$. We wish to determine the multiple solutions of Equation (3.9). The analysis proceeds as before. If

$$\tilde{\Delta} = \det \partial(h_1, h_2/m)/\partial(p,\phi)$$

then

$$\tilde{\Delta} = \alpha(1-9\rho^2/4)\cos\phi + \alpha m \sin\phi.$$

We now consider the equations $h_1 = 0$, $h_2/m = 0$, $\tilde{\Delta} = 0$ as defining functions ρ, m, α as functions of ϕ. Along the solution of these equations, we have

$$\det \frac{\partial(h_1, h_2/m, \tilde{\Delta})}{\partial(\rho, m, \alpha)} = \pm \frac{4}{3\sqrt{3}} \neq 0.$$

Therefore, we can find functions $m(\phi,\mu)$, $\alpha(\phi,\mu)$ and $\rho(\phi,\mu)$ that will satisfy the Equations (3.9) for μ close to zero and ϕ close to $\pi/2$. Eliminating ϕ from the functions $m(\phi,\mu)$ and $\alpha(\phi,\mu)$ completes the discussion of the possible bifurcation surface shown in Figure 2.

It remains to show that the surface in Figure 2 is the bifurcation surface, that is, we must show that the number of solutions (ρ,ϕ)

of Equations (2.8) changes as this surface is crossed. However, we need not check all points on the surface. It is sufficient, for example, to show that the number of solutions changes as we cross the surface in the plane $p_4 = 0$ since the only possible way for the number of solutions to change is to pass through a multiple solution. For $p_4 = 0$, Corollary 2.1 implies all solutions for $\sigma = 0$, $p_1 > 0$ and one solution for $p_1 = 0$, $\sigma \neq 0$. Therefore, the surface in Figure 2 is a bifurcation surface with the number of solutions as indicated.

For $p_2 < 0$, one can make a similar analysis as above to obtain the corresponding bifurcation surface lying below the (σ, p_4) plane. Of course, one has two more periodic solutions in each region as a result of Lemma 2.2. We do not draw the surface in this case.

We summarize these results in the following theorem.

Theorem 3.1. Let $\sigma = p_2^{1/2} p_3$. There is a neighborhood U in $\mathscr{R}^3$ of $(0,0,0)$ such that the bifurcation surface Γ for Equation (1.1) with $(p_1, \sigma, p_4) \in U$ is depicted in Figure 2 and the surface for $p_2 > 0$ is approximately given by Equation (3.8). The number of 2π-periodic solutions of Equation (1.1) at a point $(p_1, \sigma, p_4) \in U$ which are continuous in (p_3, p_4) is shown in Figure 2.

Remark 3.1. The results of this paper are easily extended to the equation

$$\frac{d^2u}{dt^2} + u = p_1 u + p_4 \frac{du}{dt} - p_2 u^3 + p_3 f(t)$$

provided that $f(t)$ is an even function of t and $\int_0^{2\pi} f(t)\cos t\, dt = \pi$. One uses the same type of analysis together with Remark 2.1.

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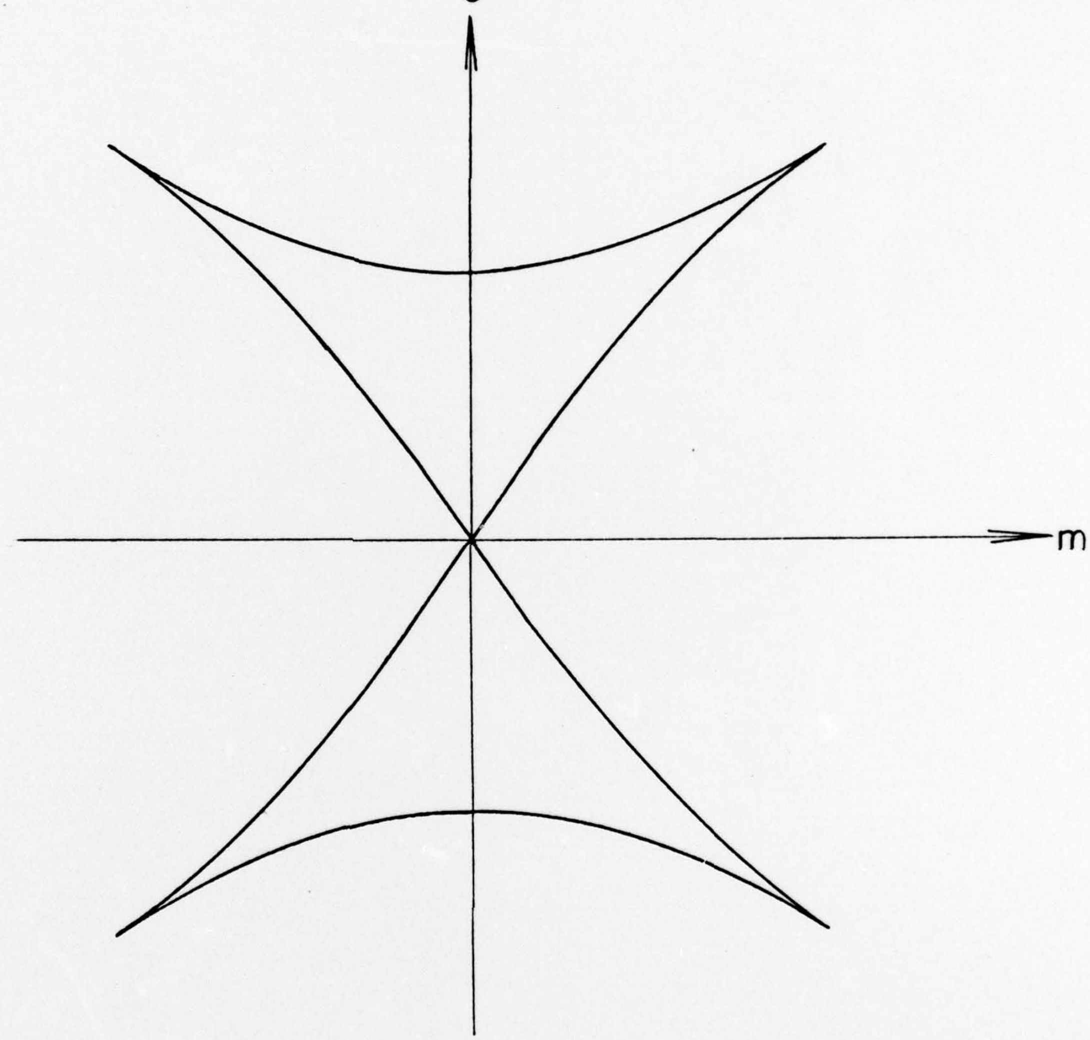
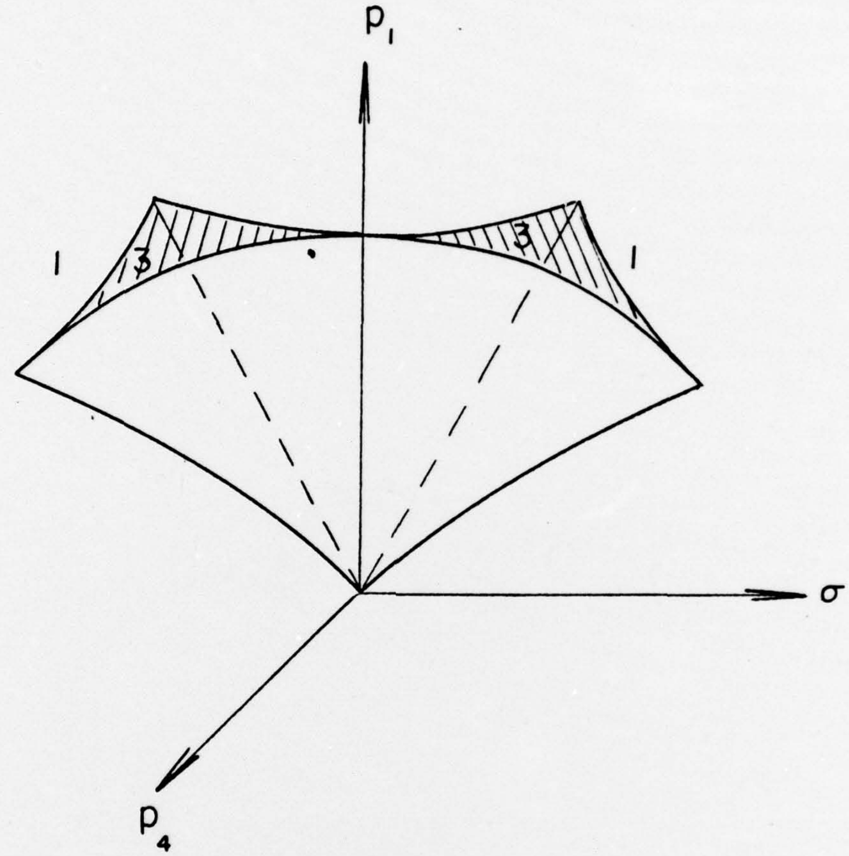


FIGURE 1



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In a previous paper, the authors gave a complete description of the number of even harmonic solutions of Duffing's equation without damping for the parameters varying in a full neighborhood of the origin in the parameter space. In this paper, the analysis is extended to the case of an independent small damping term. It is also shown that all solutions of the undamped equation are even functions of time.

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